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# Optimal Allocation under Partial Ordering of Lifetimes of Components

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# Optimal Allocation under Partial Ordering of Lifetimes of Components

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## Abstract

Assembly of systems to maximize reliability when certain components of the systems can be bolstered in different ways is an important theme in reliability Theory. This is done under assumptions of various stochastic orderings among the lifetimes of the components and the spares used to bolster them. The powerful techniques of Schur and AI functions are used in this paper to pinpoint optimal allocation results in different settings involving active and standby redundancy allocation, minimal repair and shock-threshold models.

*Key words and phrases:* Optimal Allocation, Hazard Rate Ordering, Schur and Arrangement Increasing Functions.

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AMS 1991 subject classifications. Primary 62N05; Secondary 90B25.

# 1 Introduction

Optimal assembly of a coherent system from a given set of  $n$  components is an important theme of research in reliability Theory. For instance see El-Newehi, Proschan and Sethuraman (1986). Derman, Lieberman and Ross (1974). Boland, El-Newehi and Proschan (1988) (henceforth referred to as BEP1). Boland, El-Newehi and Proschan (1992) (henceforth referred to as BEP2), Shaked and Shantikumar (1992) (henceforth referred to as SS) among others. Such an assembly may involve bolstering the original components of the system by spares which can be used as replacements upon failure (*standby redundancy*) or connected in parallel with the components (*active redundancy*). The original components can also be strengthened by performing a certain number of minimal repairs at each failure. BEP2 and SS have demonstrated the relevance of various modes of stochastic ordering of lifetimes of components in optimal allocation problems. We extend their results to more general setups using mainly the familiar techniques of AI and Schur functions.

BEP2 considered the optimal allocation of a single spare as a standby redundancy to one of the  $n$  components of a series or parallel system. In this paper we study the optimal assignment of  $n$  spares which are available as standby redundancy, one each, to the  $n$  components of series or parallel system, thus extending the results of BEP2. These results, given in Section 3, may be described informally as follows. If the components and the spares are hazard (*reverse hazard*) rate ordered and the system is a series (*parallel*) system, then one should assign the stronger (*weaker*) spares to the weaker components, in order.

We also consider the optimal allocation problem in other systems. In one class of systems, the system-lifetimes are determined by damages, introduced by shocks, exceeding random thresholds. The problem here is the optimal matching between the thresholds of the components and the parameters of the processes governing the arrival of shocks. In another class of systems, it is possible to bolster the system by performing a certain number of minimal repairs to components. When the available number of minimal repairs is specified, one has to find the optimal assignment of these minimal repairs to the various components to maximize reliability of the system. In a third class of problems, we consider a large system based on  $n$  similar modules. It is possible to bolster the first component of each module, by placing spares, in active redundancy. Given the total number of spares available, we find the optimal allocation of spares to the components. These optimal allocation results are obtained in Section 4. Some of the results here are new and others improve upon earlier ones obtained by SS and BEP1.

We collect a few definitions of various orderings among random variables and their interrelationships in Section 2. We will assume that the reader is familiar with the theory of Schur and AI functions as found for instance in the textbook of Marshall and Olkin (1979).

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## 2 Preliminaries

We will now define some of the standard concepts of ordering among random variables. In this paper we will use the words increasing to denote nondecreasing and decreasing to denote nonincreasing. Let  $X$  and  $Y$  be two random variables with distribution functions (df's)  $F$  and  $G$ , respectively. Let  $\bar{F}(x) = 1 - F(x)$  and  $\bar{G}(x) = 1 - G(x)$ . We say that

$$\begin{aligned} X &\stackrel{st}{\geq} Y \text{ if } \bar{F}(x) \geq \bar{G}(x) \text{ for all } x, \\ X &\stackrel{hr}{\geq} Y \text{ if } \bar{F}(x)/\bar{G}(x) \text{ is increasing for all } x, \\ X &\stackrel{rhr}{\geq} Y \text{ if } F(x)/G(x) \text{ is increasing for all } x, \text{ and} \\ X &\stackrel{lr}{\geq} Y \text{ if } f(x)/g(x) \text{ is increasing for all } x \end{aligned}$$

where  $f$  and  $g$  are the probability density functions (pdf's) of  $X$  and  $Y$ , respectively. It is well known, for instance see Ross (1983), that

$$\begin{aligned} X &\stackrel{lr}{\geq} Y \text{ implies that } X &\stackrel{hr}{\geq} Y \text{ implies that } X &\stackrel{st}{\geq} Y, \text{ and} \\ X &\stackrel{lr}{\geq} Y \text{ implies that } X &\stackrel{rhr}{\geq} Y \text{ implies that } X &\stackrel{st}{\geq} Y. \end{aligned}$$

When  $X$  and  $Y$  are both nonnegative random variables with absolutely continuous distributions, then saying that  $X \stackrel{hr}{\geq} Y$  ( $X \stackrel{rhr}{\geq} Y$ ) is equivalent to saying that the hazard (revers hazard) rate function of  $X$  is pointwise less than (greater than) or equal to that of  $Y$ . This explains the nomenclature for such orderings. Earlier work in this area, for instance see Keilson and Sumita (1982), have called the hazard rate ordering as uniform stochastic order in the positive direction.

Consider a unit whose lifetime has df  $F$ . The concept of minimal repair upon the failure of this unit that has been successfully used in Reliability Theory, see Barlow and Hunter (1960) and Ascher and Feingold (1984), and can be described as follows. If the item fails at time  $t$  then the minimal repair amounts to replacing the item with a functioning item of the same age, more formally, the df of the lifetime of the repaired item is given by  $(F(x) - F(t))/\bar{F}(t)$  for  $x \geq t$ . Let  $M(t)$  be the number of minimal repairs performed in time  $t$ , if minimal repairs are performed every time an item fails. It is well known that  $\{M(t), t \geq 0\}$  is a nonhomogeneous Poisson process with parameter  $-\log \bar{F}(t)$ , see Ascher and Feingold (1984).

### 3 Optimal assignment of standby redundancy to components of series and parallel systems

We begin with a brief summary of some existing work on optimal allocation of redundancy in coherent systems. Consider a system  $\mathcal{S}$  of  $n$  independent components with lifetimes  $X_1, X_2, \dots, X_n$ . Suppose that there are  $n$  independent spares with lifetimes  $Y_1, Y_2, \dots, Y_n$ . The components can be enhanced by placing the spares in parallel, one to each component that is in active redundancy. Suppose that the components and spares are ordered by the usual stochastic ordering, that is,  $X_1 \leq^{st} X_2 \leq \dots \leq^{st} X_n$  and  $Y_1 \leq^{st} Y_2 \leq \dots \leq^{st} Y_n$  and that  $\mathcal{S}$  is a  $k$ -out-of- $n$  system. BEP1 obtained fairly general results on optimal allocation in this problem. They showed that the optimal allocation is to assign spares to components in the reverse order to maximize the reliability of the system, and in fact that

$$h_{k|n}(P(X_{\pi_r} \vee Y_{\pi_r^*} > t), r = 1, 2, \dots, n)$$

is an AD function of  $(\pi, \pi^*)$ , where  $h_{k|n}(p_1, p_2, \dots, p_n)$  is the reliability function of a  $k$ -out-of- $n$  system and  $\pi$  and  $\pi^*$  are two permutations of  $(1, 2, \dots, n)$ .

Instead of using the spares in active redundancy, one can use them as replacements upon failure, that is in standby redundancy. The problem of optimal allocation of spares in this mode of redundancy has proved to be less tractable. In the recent paper, BEP2, the following setup was considered. Suppose that there is a single spare which can be assigned to some component as a replacement upon failure. Suppose that the components are ordered in the hazard rate ordering, that is  $X_1 \leq^{hr} X_2 \leq \dots \leq^{hr} X_n$  and that the system  $\mathcal{S}$  is a series system. BEP2 showed for this special case that the single spare should be assigned to the weakest component. For the case of a parallel system with components ordered by the reverse hazard ordering, they also showed that the spare should be assigned to the strongest component to maximize reliability. They also provided counterexamples to show that the hazard rate and reverse hazard rate orderings cannot be weakened to stochastic ordering and that their results do not extend to more general  $k$ -out-of- $n$  systems.

In this section we consider this situation where, instead of just one spare, there are  $n$  spares to be assigned, one each, in standby redundancy to the  $n$  components, and obtain optimal allocation results for both parallel and series systems.

The following lemmas concerning random variables ordered by the hazard rate and reverse hazard ordering are the key results in obtaining our optimal allocation results of this section.

**Lemma 1** *Let  $X, Y$  be random variables with df's  $F, G$  satisfying  $X \geq^{hr} Y$ . Let  $b(x)$  be a bounded nonnegative increasing function. Then for any bounded increasing function  $h(x)$*

$$\frac{\int h(x)b(x)dF(x)}{\int b(x)dF(x)} \geq \frac{\int h(x)b(x)dG(x)}{\int b(x)dG(x)}. \quad (3.1)$$

**Proof:** To prove the (3.1) it is enough to prove that

$$I \stackrel{\text{def}}{=} \int_{-\infty}^t b(x) dG(x) \int_{-\infty}^{\infty} b(x) dF(x) - \int_{-\infty}^t b(x) dF(x) \int_{-\infty}^{\infty} b(x) dG(x) \geq 0$$

for all  $t$  and for the class of functions  $b(x)$  of the form  $b(x) = \sum_{0 \leq i \leq m} \alpha_i I_{t_i}(x)$ , where  $\alpha_i \geq 0$ ,  $-\infty = t_0 < t_1 < t_2 < \dots < t_m < t_{m+1} = \infty$  and  $I_t(x) = 1$  if  $x > t$  and  $= 0$  if  $x \leq t$ . Fix a  $t \in (-\infty, \infty)$  such that  $t_i < t \leq t_{i+1}$  for some  $i$  with  $0 \leq i \leq m$ . In the following, we will observe the usual conventions that summations over vacuous regions are zero. It is easy to see that

$$\begin{aligned} I &= \left[ \sum_{0 \leq j \leq i} \alpha_j (\bar{G}(t_j) - \bar{G}(t)) \right] \left[ \sum_{0 \leq k \leq i} \alpha_k \bar{F}(t_k) + \sum_{i+1 \leq r \leq m} \alpha_r \bar{F}(t_r) \right] \\ &\quad - \left[ \sum_{0 \leq j \leq i} \alpha_j (\bar{F}(t_j) - \bar{F}(t)) \right] \left[ \sum_{0 \leq k \leq i} \alpha_k \bar{G}(t_k) + \sum_{i+1 \leq r \leq m} \alpha_r \bar{G}(t_r) \right] \\ &= \sum_{0 \leq j \leq i} \alpha_j^2 [\bar{F}(t) \bar{G}(t_j) - \bar{G}(t) \bar{F}(t_j)] \\ &\quad + \sum_{0 \leq j < k \leq i} \alpha_j \alpha_k [\bar{F}(t) (\bar{G}(t_j) + \bar{G}(t_k)) - \bar{G}(t) (\bar{F}(t_j) + \bar{F}(t_k))] \\ &\quad + \sum_{0 \leq j \leq i} \sum_{i+1 \leq r \leq m} \alpha_j \alpha_r [\bar{F}(t_r) (\bar{G}(t_j) - \bar{G}(t)) - \bar{G}(t_r) (\bar{F}(t_j) - \bar{F}(t))]. \end{aligned} \quad (3.2)$$

Since  $X \stackrel{hr}{\geq} Y$ , we get the two inequalities

$$\bar{F}(t) \bar{G}(t_j) - \bar{G}(t) \bar{F}(t_j) = \left[ \frac{\bar{F}(t)}{\bar{F}(t_j)} - \frac{\bar{G}(t)}{\bar{G}(t_j)} \right] \bar{F}(t_j) \bar{G}(t_j) \geq 0 \text{ when } j \leq i \quad (3.3)$$

and

$$\begin{aligned} &\bar{F}(t_r) (\bar{G}(t_j) - \bar{G}(t)) - \bar{G}(t_r) (\bar{F}(t_j) - \bar{F}(t)) \\ &= \left[ \frac{\bar{F}(t_r)}{\bar{F}(t_j)} \frac{(\bar{G}(t_j) - \bar{G}(t))}{\bar{G}(t_j)} - \frac{\bar{G}(t_r)}{\bar{G}(t_j)} \frac{(\bar{F}(t_j) - \bar{F}(t))}{\bar{F}(t_j)} \right] \bar{F}(t_j) \bar{G}(t_j) \geq 0 \text{ when } r \geq j. \end{aligned} \quad (3.4)$$

Applying the inequality (3.3) to the first two terms of (3.2) and the inequality (3.4) to the last term of (3.2), we find that  $I \geq 0$ .  $\diamond$

It is clear that (3.1) holds even when we can allow the functions  $h$  and  $b$  to be also unbounded as long as all the integrals in (3.1) are finite. For such and other generalizations see Capéraà (1988).

**Lemma 2** Let  $X_1, X_2, Y_1, Y_2$  be independent random variables such that  $X_1 \stackrel{hr}{\geq} X_2$  and  $Y_1 \stackrel{rhr}{\geq} Y_2$ . Then

$$P(X_1 \geq Y_1) P(X_2 \geq Y_2) \geq P(X_1 \geq Y_2) P(X_2 \geq Y_1). \quad (3.5)$$

**Proof:** Let the df's of  $X_1, X_2, Y_1, Y_2$  be  $F_1, F_2, G_1, G_2$ , respectively. To prove (3.5) we see that

$$\begin{aligned} & P(X_1 \geq Y_1)P(X_2 \geq Y_2) - P(X_1 \geq Y_2)P(X_2 \geq Y_1) \\ &= \int G_1(x_1)dF_1(x_1) \int G_2(x_2)dF_2(x_2) - \int G_2(x_1)dF_1(x_1) \int G_1(x_2)dF_2(x_2) \geq 0 \end{aligned}$$

by appealing to Lemma 1 and taking  $F = F_1, G = F_2, b = G_2$  and  $h = G_1/G_2$ .  $\diamond$

We now exploit the above lemma to obtain optimal allocation results for series and parallel systems when spares have to be allocated to components as standby redundancy.

**Theorem 1** Consider a system  $\mathcal{S}$  consisting of  $n$  components in series with independent lifetimes  $X_1, X_2, \dots, X_n$ . Suppose that  $n$  independent spares with lifetimes  $Y_1, Y_2, \dots, Y_n$ , which are independent of the components, are available as standby redundancy, one each to a component. Suppose that  $X_1 \stackrel{hr}{\geq} X_2 \stackrel{hr}{\geq} \dots \stackrel{hr}{\geq} X_n$  and  $Y_1 \stackrel{hr}{\geq} Y_2 \stackrel{hr}{\geq} \dots \stackrel{hr}{\geq} Y_n$ . For  $t \geq 0$ , and for permutations  $\pi, \pi^*$  of  $(1, 2, \dots, n)$ , define

$$g_t(\pi, \pi^*) = P((X_{\pi_r} + Y_{\pi_r^*}) \geq t, r = 1, \dots, n).$$

Then  $g_t(\pi, \pi^*)$  is an AD function for each  $t \geq 0$ .

**Proof:** By standard methods for AD functions it is enough to prove this result for the case  $n = 2$ . Note that  $Y_1 \stackrel{hr}{\geq} Y_2$  is equivalent to  $-Y_2 + t \stackrel{rhr}{\geq} -Y_1 + t$ , for each  $t$ . From (3.5) it follows that

$$P(X_1 + Y_2 \geq t)P(X_2 + Y_1 \geq t) \geq P(X_1 + Y_1 \geq t)P(X_2 + Y_2 \geq t).$$

This proves that  $g_t(\pi, \pi^*)$  is an AD function when  $n = 2$ .  $\diamond$

From this theorem it is clear that in order to maximize the reliability of the system  $\mathcal{S}$  one should allocate the spares with lifetimes  $Y_1, Y_2, \dots, Y_n$  to the components with lifetimes  $X_1, X_2, \dots, X_n$  in reverse order, that is the stronger spares should be allocated to the weaker components in order.

**Theorem 2** Consider a system  $\mathcal{P}$  consisting of  $n$  components in parallel with independent lifetimes  $X_1, X_2, \dots, X_n$ . Suppose that  $n$  independent spares with lifetimes  $Y_1, Y_2, \dots, Y_n$ , which are independent of the components, are available as standby redundancy, one each to a component. Suppose that  $X_1 \stackrel{rhr}{\geq} X_2 \stackrel{rhr}{\geq} \dots \stackrel{rhr}{\geq} X_n$  and  $Y_1 \stackrel{rhr}{\geq} Y_2 \stackrel{rhr}{\geq} \dots \stackrel{rhr}{\geq} Y_n$ . For  $t \geq 0$ , and for permutations  $\pi, \pi^*$  of  $(1, 2, \dots, n)$ , define

$$g_t(\pi, \pi^*) = 1 - P((X_{\pi_r} + Y_{\pi_r^*}) \leq t, r = 1, \dots, n).$$

Then  $g_t(\pi, \pi^*)$  is an AI function for each  $t \geq 0$ .



**Proof:** Again, it is enough to prove this result for the case  $n = 2$ . Note that  $Y_1 \stackrel{thr}{\geq} Y_2$  is equivalent to  $-Y_2 + t \stackrel{hr}{\geq} -Y_1 + t$ . From (3.5) it follows that

$$P(X_1 + Y_2 \leq t)P(X_2 + Y_1 \leq t) \geq P(X_1 + Y_1 \leq t)P(X_2 + Y_2 \leq t).$$

This proves that  $g_i(\pi, \pi^*)$  is an AI function when  $n = 2$ .  $\diamond$

From this theorem it is clear that in order to maximize the reliability of the system  $\mathcal{P}$  one should allocate the spares with lifetimes  $Y_1, Y_2, \dots, Y_n$  to the components with lifetimes  $X_1, X_2, \dots, X_n$  in order, that is the stronger spares should be allocated to the stronger components in order.

**Remark 1** In Theorem 1 and Theorem 2 if we take  $Y_1$  to be any nonnegative random variable and take  $Y_2 = Y_3 = \dots = Y_n = 0$  we obtain, as a special case, the results of BEP2 for the optimal allocation of a single spare mentioned earlier at the beginning of this section. BEP2 also showed that if the hazard (reverse hazard) rate ordering among the lifetimes of the components in the series system  $\mathcal{S}$  ( $\mathcal{P}$ ) were weakened to just stochastic ordering, then such an optimal allocation result will not be true. These counterexamples also show that we cannot weaken the conditions of hazard (reverse hazard) rate ordering in Theorems 1 (Theorems 2).

**Remark 2** Suppose that the series (parallel) structures considered in Theorems 1 and Theorems 2 constituted a module within a larger coherent structure. The optimal allocation conclusions of these theorems continue to hold for the larger coherent system as well.

Theorems 1 and Theorems 2 find applications when we know that the lifetimes of components and spares are ordered by the hazard and reverse hazard rate orderings. Such orderings are easily apparent if the lifetimes belong to a proportional hazards or proportional reverse hazards family. For another family of distributions see the following corollary.

**Corollary 1** Let  $U, V$  be two random variables with logconcave density functions. Let  $\mathbf{r} = (r_1, r_2, \dots, r_n)$  and  $\mathbf{s} = (s_1, s_2, \dots, s_n)$ . Suppose that the distribution of the lifetimes of the independent components and spares are given by

$$X_i = \sum_{1 \leq j \leq r_i} U_{ij}, Y_i = \sum_{1 \leq j \leq s_i} V_{ij}$$

$i = 1, 2, \dots, n$ , where the random variables  $U_{ij}$  are independent copies of  $U$  and  $V_{ij}$  are independent copies of  $V$ . Suppose that the spares are to be allocated to the components, one each, as standby redundancy. Then

$$g_i(\mathbf{r}, \mathbf{s}) \stackrel{def}{=} P((X_i + Y_i) \geq t), i = 1, \dots, n$$

is an AD function of  $(\mathbf{r}, \mathbf{s})$  and

$$h_i(\mathbf{r}, \mathbf{s}) \stackrel{def}{=} 1 - P((X_i + Y_i) \leq t), i = 1, \dots, n$$

is an AI function of  $(\mathbf{r}, \mathbf{s})$ .

**Proof:** It is well known, for instance see Karlin and Proschan (1960), that if  $U$  has a log concave density then  $X_i \stackrel{lr}{\leq} X_j$  whenever  $r_i \leq r_j$  and hence that  $X_i \stackrel{hr}{\leq} X_j$  and  $X_i \stackrel{rhr}{\leq} X_j$ . A similar conclusion holds for the lifetimes of the spares. The corollary now follows from Theorems 1 and Theorems 2.

## 4 More Applications of Schur and AI Functions in Optimal Allocation

In this section we study some further optimal allocation problems. Once again we use the standard techniques of Schur and AI functions to pinpoint the optimal allocations. It will become clear that these techniques constitute the most natural, and at the same time, very powerful tools in this area.

Consider a system  $\mathcal{ST}$  consisting of  $n$  components situated at  $n$  locations. At location  $i$ , shocks arrive according to a Poisson process  $\{N_i(t), t \geq 0\}$  with parameter  $\lambda_i$  and the  $j$ th shock produces a random damage  $D_{ij}$  to the component situated at that location, and component  $i$  dies when the total accumulated damage becomes greater than or equal to its random threshold  $X_i$ ,  $i = 1, 2, \dots, n$ . In other words

$$T_i \leq t \text{ if } \sum_{1 \leq j \leq N_i(t)} D_{ij} \geq X_i$$

where  $T_i$  is the lifetime of component  $i$ . We assume that the random variables  $\{D_{ij}\}$  are i.i.d. with a common logconcave density and that  $\{\{X_1, \dots, X_n\}, \{D_{ij}, i = 1, 2, \dots, n, j = 1, 2, \dots\}, \{N_1(t), t \geq 0\}, \dots, \{N_n(t), t \geq 0\}\}$  are independent. We will call such a system as a shock-threshold system.

The following theorem pinpoints the optimal matching between the thresholds and the parameters of the shock processes for both series and parallel systems.

**Theorem 3** Consider a shock-threshold system  $\mathcal{ST}$  as above.

(A) Suppose that  $\mathcal{ST}$  is a series system based on its components. Suppose that  $X_1 \stackrel{hr}{\leq} X_2 \stackrel{hr}{\leq} \dots \stackrel{hr}{\leq} X_n$ . Let  $T_S(\lambda)$  be the lifetime of the system  $\mathcal{ST}$ . Then for every increasing function  $g$ ,

$$E(g(T_S(\lambda))) \text{ is AI in } \lambda.$$

(B) Suppose that  $\mathcal{ST}$  is a parallel system based on its components. Suppose that  $X_1 \stackrel{rhr}{\leq} X_2 \stackrel{rhr}{\leq} \dots \stackrel{rhr}{\leq} X_n$ . Let  $T_P(\lambda)$  be the lifetime of the system  $\mathcal{ST}$ . Then for every increasing function  $g$ ,

$$E(g(T_P(\lambda))) \text{ is AD in } \lambda.$$

**Proof:** Notice that

$$\begin{aligned} P(T_S(\lambda) > t) &= P(X_i > \sum_{1 \leq j \leq N_i(t)} D_{ij}, i = 1, \dots, n) \\ &= \sum_{\mathbf{k}} P(X_i > \sum_{1 \leq j \leq k_i} D_{ij}, i = 1, \dots, n) \prod_i \exp(-\lambda_i t) \frac{(\lambda_i t)^{k_i}}{k_i!}, \end{aligned}$$

where  $\mathbf{k} = (k_1, k_2, \dots, k_n)$ . Now

$$P(X_i > \sum_{1 \leq j \leq k_i} D_{ij}, i = 1, \dots, n) = \int \dots \int P(X_1 > x_1, \dots, X_n > x_n) \prod_i f^{(k_i)}(x_i) dx_1 \dots dx_n.$$

To say that the  $X_i$ 's are hazard rate ordered is the same as saying that  $P(X_i > x)$  is  $TP_2$  in  $i$  and  $x$ . Since  $f(x)$  is logconvave,  $f^{(k)}(x)$  is  $TP_2$  in  $k$  and  $X$ , from Karlin and Proschan (1960). From Hollander, Proschan and Sethuraman (1977) products of  $TP_2$  functions are AI functions. Hence  $P(X_1 > x_1, \dots, X_n > x_n)$  is AI in  $\mathbf{x}$  and  $\prod_i f^{(k_i)}(x_i)$  is AI in  $(\mathbf{x}, \mathbf{k})$ . Thus from the preservation theorem Hollander, Proschan and Sethuraman (1977) (Theorem 3.3)  $P(X_i > \sum_{1 \leq j \leq k_i} D_{ij}, i = 1, \dots, n)$  is AI in  $\mathbf{k}$ . Again  $\prod_i \exp -\lambda_i t \frac{(\lambda_i t)^{k_i}}{k_i!}$  is AI in  $(\lambda, \mathbf{k})$ . By invoking the same preservation theorem we obtain that  $P(T_S(\lambda) > t)$  is AI in  $\lambda$ . This proves part (A) pertaining to a series system. Part (B) is proved in a similar fashion.  $\diamond$

Suppose that in a shock-threshold system described above we could permute the components amongst the locations, still putting only one component at each location. The above theorem shows that if the system is a series system based on its components then we should allocate the weaker components to the locations with lesser intensity of shocks, in order, to stochastically maximize the lifetime of the system. If the system were a parallel system based on its components, then we should allocate the stronger components to the locations with lesser intensity of shocks, in order, to stochastically maximize the lifetime of the system. This agrees with the intuitive notions regarding series and parallel systems.

We will now consider a system  $\mathcal{S}$  with  $n$  components and suppose that we can bolster the components by performing minimal repair upon failure. SS considered this problem when the lifetimes of the components are ordered stochastically and there is a vector  $(k_1, k_2, \dots, k_n)$  representing the numbers of minimal repairs that can be performed on the components. When  $\mathcal{S}$  is a parallel system they showed that the optimal allocation is to assign smaller  $k_r$ 's to the weaker components, in order. In what follows we provide a simpler proof of this result and give a similar result when  $\mathcal{S}$  is a series system.

**Theorem 4** Suppose that  $X_1 \stackrel{st}{\leq} X_2 \stackrel{s}{\leq} \dots \stackrel{st}{\leq} X_n$ , with df's  $F_1, F_2, \dots, F_n$ , are the independent lifetimes of components 1, 2, ...,  $n$ . Let  $\mathbf{k} = (k_1, \dots, k_n)$  denote the number of minimal repairs assigned to components 1, ...,  $n$ , respectively. Let  $\bar{H}_P(t, \mathbf{k})$  ( $\bar{H}_S(t, \mathbf{k})$ ) be the survival distribution of a parallel (series) system formed of components 1, ...,  $n$ . Then

- (i)  $\bar{H}_P(t, \mathbf{k})$  is AI in  $\mathbf{k}$  for each  $t \geq 0$ .
- (ii)  $\bar{H}_S(t, \mathbf{k})$  is AD in  $\mathbf{k}$  for each  $t \geq 0$ .

**Proof:** From the standard techniques used to prove that functions are AI or AD, we can assume that  $n = 2$ . Let  $M_i(t)$  be the number of minimal repairs performed in time  $t$  on component  $i$ ,  $i = 1, 2$  assuming that we can perform an unlimited number of repairs. Then

$$E(M_1(t)) = -\log(1 - F_1(t)) \geq -\log(1 - F_2(t)) = E(M_2(t)).$$

Let  $X_i(k)$  be the lifetime of the  $i$ th component when  $k$  minimal repairs are performed on it. Let  $k_1 \leq k_2$ . Since the Poisson distribution is likelihood ratio ordered in its parameter, it is also hazard and reverse hazard rate ordered in its parameter. Thus

$$P(M_1(t) \geq k_1 + 1)P(M_2(t) \geq k_2 + 1) \leq P(M_1(t) \geq k_2 + 1)P(M_2(t) \geq k_1 + 1).$$

Now

$$\begin{aligned} \bar{H}_{\mathcal{P}}(t, k_1, k_2) &= 1 - P(M_1(t) \geq k_1 + 1, M_2(k_2) \geq k_2 + 1) \\ &\geq 1 - P(M_1(t) \geq k_2 + 1, M_2(k_2) \geq k_1 + 1) \\ &= 1 - P(X_1(k_2) \leq t, X_2(k_1) \leq t) = \bar{H}_{\mathcal{P}}(t, k_2, k_1). \end{aligned}$$

The second part is established in a similar fashion by using the fact that the Poisson processes  $M_1(t)$  and  $M_2(t)$  are reverse hazard ordered.  $\diamond$

**Remark 3** In the above theorem, if we were given a vector  $\mathbf{k}$  with  $k_1 \leq k_2 \leq \dots \leq k_n$  and we could choose a permutation  $\pi$  and allocate  $k_{\pi_i}$  minimal repairs to components  $i$ ,  $i = 1, 2, \dots, n$ . The optimal allocation that maximizes stochastically the lifetime of a parallel (series) system based on these components is to choose  $\pi = (1, 2, \dots, n)$  ( $\pi = (n, n-1, \dots, 1)$ ). The first part of the above theorem is due to SS. Our proof here is shorter and our use of AI functions allowed us to extend the results to the series case.

We end this section by considering a situation of optimal allocation in which specific components within modules of a system are targeted for bolstering by means of active redundancy. We describe some variants of this problem that have been considered before and then present our extension.

Consider  $n$  components  $C_1, C_2, \dots, C_n$  which can be bolstered by adding spares in parallel, that is in active redundancy. Suppose that we have a vector  $(k_1, k_2, \dots, k_n)$  representing the number of spares which are available for active redundancy. When the components and spares are identically distributed and the system is a  $k$ -out-of- $n$  system, BEP1 showed that the reliability of the system is Schur-concave in  $(k_1, k_2, \dots, k_n)$ . When these components are parts of  $n$  similar subsystems and the larger system is a series system based on these subsystems, SS showed that the reliability of the supersystem is Schur-concave in  $(k_1, k_2, \dots, k_n)$ . Such results immediately pinpoint the optimal choice of  $(k_1, k_2, \dots, k_n)$ . We now generalize both these results to the case where the components are parts of  $n$  similar subsystems and the larger system is a  $m$ -out-of- $n$  system based on these  $n$  subsystems.

**Theorem 5** Let  $P$  be a coherent system with  $N$  components. Let the first component of  $P$  be independent of the rest of the components. Let  $P_1, P_2, \dots, P_n$  be  $n$  independent copies of  $P$ . Let  $C_i$  be the first component of  $P_i$  and suppose that we wish to bolster  $C_i$  with  $k_i$  independent spares in active redundancy,  $i = 1, \dots, n$ . Suppose that the spares and the components  $C_1, \dots, C_n$  have lifetimes with common cdf  $F$ . Consider a larger system  $S$  which is an  $m$ -out-of- $n$  system based on  $P_1, P_2, \dots, P_n$ . Let  $T(\mathbf{k})$  be the lifetime of  $S$ . Then  $E(g(T(\mathbf{k})))$  is Schur concave in  $\mathbf{k}$ , for every increasing function  $g$ .

**Proof:** Let  $T_i(k_i)$  be the lifetime of  $P_i$  when its first component  $C_i$  has been bolstered with  $k_i$  spares in active redundancy. Note that  $P(T_i(k_i) \leq t) = a + bF^{k_i+1}(t)$  where  $a, b$  are nonnegative constants related to the reliabilities of the system  $P$  with a functioning or nonfunctioning component 1, and hence do not depend on  $i$ . Thus

$$P(T(\mathbf{k}) > t) = 1 - h_{m-n+1|n}(a + bF^{k_1+1}(t), \dots, a + bF^{k_n+1}(t))$$

where  $h_{m-n+1|n}(p_1, \dots, p_n)$  is the reliability function of an  $m-n+1$ -out-of- $n$  system based independent components with reliabilities  $p_1, \dots, p_n$ . Pledger and Proschan (1971) showed that  $h_{m-n+1|n}(p_1, \dots, p_n)$  is Schur convex in  $(\log p_1, \dots, \log p_n)$  and is clearly an increasing function of its arguments. Let  $\mathbf{k} \geq^m \mathbf{k}'$ . Since  $F(t)^{k+1}$  is convex in  $k$ , it follows that  $F(t)^{k_1+1} + F(t)^{k_2+1}$  is Schur convex in  $(k_1, k_2)$ . We can use this fact to show that  $(\log(a + bF^{k_1+1}(t)), \dots, \log(a + bF^{k_n+1}(t))) \geq^{wm} (\log(a + bF^{k'_1+1}(t)), \dots, \log(a + bF^{k'_n+1}(t)))$ . Hence  $P(T(\mathbf{k}) > t) \leq P(T(\mathbf{k}') > t)$  for each  $t$ . Thus  $E(g(T(\mathbf{k})))$  is Schur concave in  $\mathbf{k}$ , for every increasing function  $g$   $\diamond$

The above theorem generalizes previous results of BEP1 and SS. BEP1 obtained the above result for the case where  $P$  consisted of only one component. For more general systems  $P$ , SS obtained the above result when the larger system is a series system based on  $P_1, \dots, P_n$ , that is when  $m = n$ .

**Remark 4** Suppose that a total of  $L$  spares is available. Since  $E(g(T(\mathbf{k})))$  is Schur concave in  $\mathbf{k}$  from the above theorem, we have to distribute the spares in the most homogeneous fashion to obtain the optimal allocation. In fact, the optimal allocation vector is given by  $\mathbf{k} = (\underbrace{s+1, \dots, s+1}_r, \underbrace{s, \dots, s}_{n-r})$  where  $s, r$  are determined uniquely by the equation  $L = sn + r, 0 \leq r \leq n-1$ .

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